

# Bi-Hamiltonian Structures of Chaotic Dynamical Systems in 3D

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**Abstract:** We study Poisson structures of dynamical systems with three degrees of freedom which are known for their chaotic properties, namely Lü, modified Lü, Chen,  $T$  and Qi systems. We show that all these flows admit bi-Hamiltonian structures depending on the values of their parameters.

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**Key words:** First integrals, Poisson structures, Jacobi's last multiplier, chaotic three dimensional systems, bi-Hamiltonian structures, Nambu-Poisson bracket.

## 1 Introduction

In 1963, Lorenz discovered the first (autonomous and three dimensional) chaotic dynamical system [25]. Fifteen years later, Rössler constructed a more simple chaotic system involving only one nonlinear term [33]. More recently, in 1999, Chen and Ueta defined three dimensional chaotic Chen system as a dual of Lorenz system using chaotification approach [4, 38]. Three years after the presentation of the Chen system, Lü system was introduced [26, 27]. Lü system is an intermediate system that bridges the gap between the Chen and the Lorenz systems. In 2005, chaotic Qi system was presented by completing quadratic cross product terms some of which are missing in the Chen system [32]. In 2008, Tigan presented chaotic  $T$  system which is structurally looks like the same with the Lorenz system except a linear term [37].

Even though the association of chaotic behaviour with integrability had not generally been expected, many extensive studies have been done on the Hamiltonian structures of chaotic dynamics. Two of these studies, [13] and [8], are important for the present paper. In [13], Gümral and Nutku showed that every Hamiltonian system in three dimensions is mutually bi-Hamiltonian and derived non-canonical bi-Hamiltonian structure of the Lorenz system for certain values of its parameters. In [8], Gao showed that the necessary and sufficient condition of a three dimensional dynamical system  $\dot{\mathbf{x}} = \mathbf{X}$  (having a time independent first integral) being Hamiltonian is existence of a non-zero function  $M$ , called Jacobi's last multiplier, which makes  $M\mathbf{X}$  divergence free. Using this, Gao achieved to present Hamiltonian structure of the Lotka-Volterra system. In the present work, motivating by these observations and following historical order presented in the previous

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paragraph, we shall investigate the Hamiltonian structures of three dimensional chaotic Lü, modified Lü, Chen, Qi and  $T$  systems. We shall be able to show that all these systems are bi-Hamiltonian for certain values of their parameters.

This paper consists of two main sections. In order to make the paper self-contained, next section will be started with the definitions of Poisson and Nambu-Poisson brackets, and presentations of Hamiltonian, bi-Hamiltonian and Nambu-Hamiltonian systems. We shall point out three important properties of the Hamiltonian structures in dimension three. The first one is that they always come in compatible pairs to form a bi-Hamiltonian structure [13]. The second one is that existence of a Jacobi's last multiplier  $M$  completely determines existence of bi-Hamiltonian structure of a dynamical system [8]. The third one is that, in three dimensions, a Hamiltonian (hence a bi-Hamiltonian) system is mutually Nambu-Hamiltonian.

In section 3, we shall focus on three-dimensional autonomous Lü, modified Lü, Chen, Qi and  $T$  systems. One common feature of these chaotic systems is that, for some certain values of their parameters, they can be transformed in such a way that the generating vector fields become divergence free. Hence, Jacobi's last multipliers of all these systems can be taken as the unity. This enables us to derive bi-Hamiltonian structures of them after the presentation of a time independent first integral. We remark that, for all these systems, except the Lü system, one of the Poisson structures is time-dependent.

This work may be considered as a complementary to [6], where bi-Hamiltonian structures and superintegrability of some four dimensional hyperchaotic systems were studied.

## 2 Bi-Hamiltonian structure of 3D systems

### 2.1 Poisson brackets

A first integral of a dynamical system is a real-valued function that retains constant values on integral curves of the system. An  $n$ -dimensional system is maximally superintegrable if it admits  $n - 1$  first integrals [15, 36]. Note that, existence of  $n - 1$  first integrals lets one to reduce the systems of differential equations to a one quadrature. In dimension three, two first integrals are needed for (maximal) superintegrability.

Poisson bracket on an  $n$ -dimensional space is a binary operation  $\{\bullet, \bullet\}$  on the space of real-valued smooth functions satisfying the Leibnitz and the Jacobi identities [23, 24, 31, 39]. We define a Poisson bracket of two functions  $F$  and  $H$  by

$$\{F, H\} = \nabla F \cdot N \nabla H, \quad (1)$$

where  $N$  is skew-symmetric Poisson matrix,  $\nabla F$  and  $\nabla H$  are gradients of  $F$  and  $H$ , respectively. Poisson bracket (1) automatically satisfies the Leibnitz identity, whereas, in a local frame  $(x^i)$ , the Jacobi identity turns out to be

$$N^{i[j} \partial_{x^i} N^{kl]} = 0 \quad (2)$$

assuming summation over repeated indices. In Eq.(2),  $N^{ij}$  are components of the Poisson matrix  $N$ , superscript bracket  $[ \ ]$  refers anti-symmetrization, and  $\partial_{x^i}$  denotes the partial differentiation with respect to  $x^i$ . A Casimir function  $C$  on a Poisson space is the one that commutes with all the other functions. In order to have a non-trivial Casimir function, the Poisson matrix  $N$  must be degenerate.

A system of ODE's is Hamiltonian if it can be written in the form of Hamilton's equation

$$\dot{x}^i = \{x^i, H\}, \quad i = 1, \dots, n, \quad (3)$$

for  $H$  being a real-valued function, called Hamiltonian function, and  $\{\bullet, \bullet\}$  being a Poisson bracket. For the Poisson bracket (1), Hamilton's equation takes the particular form

$$\dot{\mathbf{x}} = N \nabla H. \quad (4)$$

In order to express a finite-dimensional dynamical system in the form of Hamilton's equation, two tasks must be accomplished. One is to define a proper Poisson matrix and the other is to find a proper Hamiltonian function. When searching a Poisson matrix, main hurdle is to find a solution of the Jacobi identity (2). For three dimensional flows, this can be computed relatively simply since the Jacobi identity yields a scalar equation. On the other hand, in order to find a Hamiltonian function, one may try to find the first integrals of the system, because it is often observed that a first integral or some function of the first integral(s) becomes the Hamiltonian function. Note that, for autonomous systems, in case of the existences, both of the Hamiltonian and the Casimir functions are the first integrals of the system [15, 31], whereas, for the case of non-autonomous systems, they may depend explicitly on time so that they may fail to be integral invariants [1].

A dynamical system is bi-Hamiltonian if it admits two different Hamiltonian structures

$$\dot{\mathbf{x}} = N_1 \nabla H_2 = N_2 \nabla H_1, \quad (5)$$

with the requirement that the Poisson matrices  $N_1$  and  $N_2$  be compatible, that is any linear pencil  $N_1 + cN_2$  must satisfy the Jacobi identity (2), [22, 31]. In three dimensions, an autonomous bi-Hamiltonian system is superintegrable. We refer [9] for the multi-Hamiltonian structures of the maximal superintegrable systems of arbitrary order.

## 2.2 Nambu-Poisson brackets

In [28], a ternary operation  $\{\bullet, \bullet, \bullet\}$ , called Nambu-Poisson bracket, is defined on the space of smooth functions satisfying the generalized Leibnitz identity

$$\{F_1, F_2, FH\} = \{F_1, F_2, F\}H + F\{F_1, F_2, H\} \quad (6)$$

and the fundamental (or Takhtajan) identity

$$\{F_1, F_2, \{H_1, H_2, H_3\}\} = \sum_{k=1}^3 \{H_1, \dots, H_{k-1}, \{F_1, F_2, H_k\}, H_{k+1}, \dots, H_3\}, \quad (7)$$

for arbitrary functions  $F, F_1, F_2, H, H_1, H_2$ . A dynamical system is called Nambu-Hamiltonian with Hamiltonian functions  $H_1$  and  $H_2$  if it can be recasted as

$$\dot{x}^i = \{x^i, H_1, H_2\}. \quad (8)$$

By fixing the Hamiltonian functions  $H_1$  and  $H_2$ , we can write Nambu-Hamiltonian system (8) in the bi-Hamiltonian form

$$\dot{x}^i = \{x^i, H_1\}^{H_2} = \{x^i, H_2\}^{H_1} \quad (9)$$

where the Poisson brackets  $\{\bullet, \bullet\}^{H_2}$  and  $\{\bullet, \bullet\}^{H_1}$  are defined by

$$\{F, H\}^{H_2} = \{F, H, H_2\}, \quad \{F, H\}^{H_1} = \{F, H_1, H\}, \quad (10)$$

respectively. Both of the brackets  $\{\bullet, \bullet\}^{H_2}$  and  $\{\bullet, \bullet\}^{H_1}$  satisfy the Jacobi and Leibnitz identities [12]. In order to show that they constitute a compatible Poisson pair we need

to confirm that an arbitrary linear pencil  $\{\bullet, \bullet\}_{LP} = \{\bullet, \bullet\}^{H_2} + \lambda\{\bullet, \bullet\}^{H_1}$  is satisfying the Jacobi identity. To establish this, we compute

$$\begin{aligned}
& \bigcirc_{H,F,K} \{\{H, F\}_{LP}, K\}_{LP} \\
&= \bigcirc_{H,F,K} (\{\{H, F\}^{H_2} + \lambda\{H, F\}^{H_1}, K\}^{H_2} + \lambda\{\{H, F\}^{H_2} + \lambda\{H, F\}^{H_1}, K\}^{H_1}) \\
&= \bigcirc_{H,F,K} (\{\{H, F\}^{H_2}, K\}^{H_2} + \lambda\{\{H, F\}^{H_1}, K\}^{H_2} + \lambda\{\{H, F\}^{H_2}, K\}^{H_1} \\
&\quad + \lambda^2\{\{H, F\}^{H_1}, K\}^{H_1}) \\
&= \lambda \bigcirc_{H,F,K} (\{\{H, F\}^{H_1}, K\}^{H_2} + \{\{H, F\}^{H_2}, K\}^{H_1}) \\
&= \lambda \bigcirc_{H,F,K} (\{\{H, H_1, F\}, K, H_2\} + \{\{H, F, H_2\}, H_1, K\}), \tag{11}
\end{aligned}$$

where  $\bigcirc$  denotes the sum over the cyclic permutation of  $H, F$  and  $K$  while keeping  $H_1$  and  $H_2$  constants. In the calculation, the first and the fourth terms in second line vanish since the brackets  $\{\bullet, \bullet\}^{H_2}$  and  $\{\bullet, \bullet\}^{H_1}$  satisfy the Jacobi identity. In the fourth line, the definitions in Eq.(10) are used. In Eq.(11), expansion of the summation, reordering the arguments and application of the Takhtajan's identity (7) result with

$$\begin{aligned}
& \lambda\{\{H, H_1, F\}, K, H_2\} + \lambda\{\{F, H_1, K\}, H, H_2\} + \lambda\{\{K, H_1, H\}, F, H_2\} \\
&\quad + \{\{H, F, H_2\}, H_1, K\} + \{\{F, K, H_2\}, H_1, H\} + \{\{K, H, H_2\}, H_1, F\} \\
&= -\lambda\{\{F, H_1, H\}, K, H_2\} - \lambda\{H, \{F, H_1, K\}, H_2\} - \lambda\{\{K, H, H_1\}, F, H_2\} \\
&\quad - \lambda\{H_1, \{F, H_2, H\}, K\} - \lambda\{H_1, H, \{F, H_2, K\}\} + \lambda\{\{K, H, H_2\}, F, H_1\} \\
&= -\lambda\{F, H_1, \{H, K, H_2\}\} + \lambda\{H, K, \{F, H_1, H_2\}\} - \lambda\{\{K, H, H_1\}, F, H_2\} \\
&\quad - \lambda\{F, H_2, \{H_1, H, K\}\} + \lambda\{\{F, H_2, H_1\}, H, K\} + \lambda\{\{K, H, H_2\}, F, H_1\} \\
&= 0. \tag{12}
\end{aligned}$$

That is,  $\{\bullet, \bullet\}_{LP}$  is a Poisson bracket, hence  $\{\bullet, \bullet\}^{H_2}$  and  $\{\bullet, \bullet\}^{H_1}$  are compatible. Inversely, in dimension three, expressing a bi-Hamiltonian dynamics in the form of a Nambu-Hamiltonian system is possible as we shall show in the next section. For a general discussion on the linear pencils of Poisson brackets, we refer [3] which is, additionally, showing how Casimir functions of the pencil yield pairwise commuting functions with respect to both of the brackets defining the pencil.

A generalization of Nambu-Poisson bracket was presented in [34] by introducing an  $r$ -ary operation, called generalized Nambu bracket, for  $r = 2, 3, \dots$ . This generalization covers both of the Poisson and the Nambu-Poisson brackets, that is, it reduces to a Poisson bracket when  $r = 2$ , and it turns out to be a classical Nambu bracket when  $r = 3$ . A system is generalized Nambu-Hamiltonian if it can be written as

$$\dot{x}^i = \{x^i, H_1, H_2, \dots, H_{r-1}\} \tag{13}$$

under the existence of  $r - 1$  number of Hamiltonian functions. Note that, by generalizing the procedure described in Eq.(9), a generalized Nambu-Hamiltonian system can be recasted as a multi-Hamiltonian system. For a detailed discussion on the relationship between generalized Nambu-Poisson and Poisson brackets, we refer [12, 35]. In [12], additionally, investigations on quadratic Poisson structures containing celebrated Sklyanin algebras can be found.

### 2.3 Bi-Hamiltonian structures in three dimensions

Space of three dimensional vectors and space of three by three skew-symmetric matrices are isomorphic via the map

$$\mathbf{J} = (X, Y, Z) \longleftrightarrow N = \begin{pmatrix} 0 & -Z & Y \\ Z & 0 & -X \\ -Y & X & 0 \end{pmatrix}. \quad (14)$$

The isomorphism (14) can be realized by defining the identity  $N\mathbf{B} = \mathbf{J} \times \mathbf{B}$  for  $\mathbf{B}$  being an arbitrary vector. Existence of this isomorphism enables us to identify a three by three Poisson matrix  $N$  with a three dimensional Poisson vector field  $\mathbf{J}$ . In this case, the Jacobi identity (2) turns out to be

$$\mathbf{J} \cdot (\nabla \times \mathbf{J}) = 0, \quad (15)$$

see, for example, [14].  $C$  is a Casimir function if its gradient  $\nabla C$  is parallel to  $\mathbf{J}$  that is

$$\mathbf{J} \times \nabla C = \mathbf{0}. \quad (16)$$

Under the isomorphism (14), in three dimensions, Hamilton's equation takes the particular form

$$\dot{\mathbf{x}} = \mathbf{J} \times \nabla H, \quad (17)$$

whereas a bi-Hamiltonian system is in form

$$\dot{\mathbf{x}} = \mathbf{J}_1 \times \nabla H_2 = \mathbf{J}_2 \times \nabla H_1. \quad (18)$$

Note that, in order to guarantee the compatibility, Poisson vector fields  $\mathbf{J}_1$  and  $\mathbf{J}_2$  must satisfy

$$\mathbf{J}_1 \cdot \nabla \times \mathbf{J}_2 + \mathbf{J}_2 \cdot \nabla \times \mathbf{J}_1 = 0. \quad (19)$$

General solution of the Jacobi identity (15) is

$$\mathbf{J} = \frac{1}{M} \nabla H_1 \quad (20)$$

for arbitrary functions  $M$  and  $H_1$  [2, 16, 17, 18]. Existence of the scalar multiple  $1/M$  in the solution is a manifestation of the conformal invariance of Jacobi identity. In the literature,  $M$  is called Jacobi's last multiplier [15, 20, 21, 40]. The potential function  $H_1$  in Eq.(20) is a Casimir function of the Poisson vector field  $\mathbf{J}$ . Any other Casimir of  $\mathbf{J}$  has to be linearly dependent to the potential function  $H_1$  since the kernel of the equation (16) is one dimensional. Substitution of the general solution (20) of  $\mathbf{J}$  into the Hamilton's equation (17) results with

$$\dot{\mathbf{x}} = \frac{1}{M} \nabla H_1 \times \nabla H_2. \quad (21)$$

This shows that velocity vector  $\dot{\mathbf{x}}$  is orthogonal to the gradient functions, that is the solution curve  $\mathbf{x}$  is parallel to the intersection of level surfaces of  $H_1$  and  $H_2$  [11]. In the rest of this subsection, we investigate geometry of the Hamilton's equation (21) in detail.

The first observation is as follows. We choose two Poisson vector fields

$$\mathbf{J}_1 = \frac{1}{M} \nabla H_1, \quad \mathbf{J}_2 = -\frac{1}{M} \nabla H_2. \quad (22)$$

Substitution of the Poisson vector fields in Eq.(22) into the compatibility condition (19) results with

$$\begin{aligned}
& -\frac{1}{M}\nabla H_1 \cdot \nabla \times \left(\frac{1}{M}\nabla H_2\right) - \frac{1}{M}\nabla H_2 \cdot \nabla \times \frac{1}{M}\nabla H_1 \\
& = -\frac{1}{M^2}\nabla H_1 \cdot (\nabla \times \nabla H_2) + \frac{1}{M^3}\nabla H_1 \cdot (\nabla M \times \nabla H_2) - \frac{1}{M^2}\nabla H_2 \cdot (\nabla \times \nabla H_1) \\
& \quad + \frac{1}{M^3}\nabla H_2 \cdot (\nabla M \times \nabla H_1) \\
& = 0,
\end{aligned} \tag{23}$$

where the skew-symmetry of the triple product is used. So that, Poisson vector fields in Eq.(22) are compatible and Hamilton's equation (21) is bi-Hamiltonian in the form of Eq.(18). This shows that Hamiltonian structures of dynamical systems, in three dimensions, always come in compatible pairs to form a bi-Hamiltonian structure [13].

The second observation can be stated as follows. A three dimensional dynamical system  $\dot{\mathbf{x}} = \mathbf{X}$  having a time independent first integral is bi-Hamiltonian if and only if there exist a Jacobi's last multiplier  $M$  which makes  $M\mathbf{X}$  divergence free [8]. Sketch of the proof is as follows. For a bi-Hamiltonian dynamical system  $\dot{\mathbf{x}} = \mathbf{X}$  in the form of Eq.(21), scalar multiple  $M\mathbf{X}$  of the vector field  $\mathbf{X}$  equals to the cross product  $\nabla H_1 \times \nabla H_2$  of two gradients. Hence, the divergence of  $M\mathbf{X}$  vanishes, that is

$$\nabla \cdot (M\mathbf{X}) = 0. \tag{24}$$

Inversely, assume that a dynamical system  $\dot{\mathbf{x}} = \mathbf{X}$  (not necessarily Hamiltonian) has a first integral  $C$ . Then the gradient  $\nabla C$  is parallel to the vector field  $\mathbf{X}$  hence parallel to any of its scalar multiple, say  $M\mathbf{X}$ . There are infinitely many vectors satisfying

$$M\mathbf{X} = \nabla C \times \mathbf{A}. \tag{25}$$

We choose the vector field  $\mathbf{A}$  so that angle between the curl  $\nabla \times \mathbf{A}$  and  $\nabla C$  be different than  $\pi/2$ . The divergence condition (24) becomes

$$\begin{aligned}
0 &= \nabla \cdot (M\mathbf{X}) = \nabla \cdot (\nabla C \times \mathbf{A}) \\
&= \nabla \times \nabla C + \nabla C \cdot \nabla \times \mathbf{A} \\
&= \nabla C \cdot \nabla \times \mathbf{A}.
\end{aligned} \tag{26}$$

This is valid only if the curl of  $\mathbf{A}$  vanishes. Thus,  $\mathbf{A}$  locally equals to a gradient vector field  $\nabla H$  for some function  $H$ . Substitution of  $\nabla H$  in Eq.(25) enables us to recast  $\dot{\mathbf{x}} = \mathbf{X}$  in the form of Eq.(21). For the first Hamiltonian formulation, we take  $H$  as the Hamiltonian function and  $(1/M)\nabla C$  as the Poisson vector field. For the second Hamiltonian formulation, we take  $C$  as the Hamiltonian function and  $-(1/M)\nabla H$  as the Poisson vector.

When, particularly, a system  $\dot{\mathbf{x}} = \mathbf{X}$  is divergence-free, then any constant function can be taken as the Jacobi's last multiplier. In this case, finding a time independent first integral is sufficient to conclude that the system is bi-Hamiltonian. For the use of method of Jacobi's last multiplier to determine Lagrangian picture of dynamical systems, we refer [10, 29, 30].

The third observations is as follows. On the three dimensional space, we define a bracket of three functions  $F$ ,  $H_1$  and  $H_2$  as the triple product

$$\{F, H_1, H_2\} = \frac{1}{M}\nabla F \cdot \nabla H_1 \times \nabla H_2 \tag{27}$$

of their gradient vectors. It is immediate to see that the bracket in Eq.(27) satisfies both of the generalized Leibnitz identity (6) and the fundamental identity (7), so that it is a Nambu-Poisson bracket. Note that, the Hamilton's equation (21) is Nambu-Hamiltonian (8) with the bracket (27) having the Hamiltonian functions  $H_1$  and  $H_2$ . Combining this observation with the first one, we conclude that, a Hamiltonian (hence a bi-Hamiltonian) dynamics in three dimensions is mutually Nambu-Hamiltonian whereas combining with the second one, we conclude that, a system is Nambu-Hamiltonian if and only if there exists a Jacobi's last multiplier satisfying the divergence condition in Eq.(24).

### 3 Examples

#### 3.1 Lü system

Chaotic Lü system consists of three autonomous first order differential equations

$$\begin{aligned}\dot{x} &= \alpha(y - x), \\ \dot{y} &= \gamma y - xz, \\ \dot{z} &= xy - \beta z,\end{aligned}\tag{28}$$

where  $\alpha, \beta$  and  $\gamma$  are real constant parameters [26, 27]. We, particularly, take  $\beta = -\gamma = 2\alpha$  and change the dependent variables  $(x, y, z)$  to  $(u, v, w)$  according to

$$u = xe^{\alpha t}, \quad v = ye^{-\gamma t}, \quad w = ze^{\beta t}.\tag{29}$$

Additionally, we rescale the time variable by  $\bar{t} = -e^{-\alpha t}/\alpha$ . Then Lü system (28) turns out to be

$$u' = \alpha v, \quad v' = -uw, \quad w' = -uv,\tag{30}$$

where, prime denotes derivative with respect to the new time parameter  $\bar{t}$ .

The system (30) is divergence free, hence for the satisfaction of the PDE in (24), Jacobi's last multiplier  $M$  must a constant function. In this case, the system (30) admits two time independent first integrals

$$H_1(u, v, w) = \frac{1}{2}(v^2 + w^2), \quad H_2(u, v, w) = \frac{1}{2}u^2 - \alpha w.\tag{31}$$

The definitions of Poisson vectors in Eq.(22) enables us to arrive

$$\mathbf{J}_1 = \nabla H_1 = (0, v, w), \quad \mathbf{J}_2 = -\nabla H_2 = (-u, 0, \alpha).\tag{32}$$

So that, the system (30) exhibits a bi-Hamiltonian structure in form of Eq.(18) by choosing Hamiltonian functions  $H_1$  and  $H_2$  as in Eq.(31) and choosing Poisson vectors  $\mathbf{J}_1$  and  $\mathbf{J}_2$  as in Eq.(32), respectively.

#### 3.2 A modified Lü system

We add a cross product term  $yz$  to the right hand side of the first equation in the Lü system (28) and obtain

$$\begin{aligned}\dot{x} &= \alpha(y - x) + yz, \\ \dot{y} &= \gamma y - xz, \\ \dot{z} &= xy - \beta z,\end{aligned}\tag{33}$$

which is called modified Lü system, see [7]. We take  $\beta = 2\alpha$  and  $\gamma = \alpha$ , then apply a change of variables

$$u = xe^{\alpha t}, \quad v = ye^{-\alpha t}, \quad w = ze^{2\alpha t}, \quad (34)$$

which results with a non-autonomous system

$$\dot{u} = \alpha v + vwe^{-2\alpha t}, \quad \dot{v} = -uwe^{-2\alpha t}, \quad \dot{w} = uv, \quad (35)$$

admitting a time independent first integral

$$H_1(u, v, w) = \frac{1}{2}(u^2 + v^2) - \alpha w. \quad (36)$$

The system (35) is divergence free hence we take Jacobi' last multiplier as  $M = 1$  and arrive the following Poisson vector field

$$\mathbf{J}_1 = \nabla H_1 = (u, v, -\alpha). \quad (37)$$

with Hamiltonian function

$$H_2 = -\frac{v^2}{2} + \frac{e^{2\alpha t}}{2\alpha^2}((u^2 + v^2)(\frac{1}{4}(u^2 + v^2) - \alpha w)). \quad (38)$$

Note that,  $H_2$  is time dependent hence it is not an integral invariant of the system (35). Using  $H_2$ , we construct a time dependent Poisson matrix

$$\mathbf{J}_2 = -\nabla H_2 = \left( -\frac{e^{2\alpha t}u}{\alpha^2}(u^2 + v^2 - \alpha v), v - \frac{e^{2\alpha t}v}{\alpha^2}(v^2 + u^2 - \alpha u), \frac{e^{2\alpha t}}{2\alpha}(u^2 + v^2) \right). \quad (39)$$

As a result, we proved that the modified Lü system (35) is bi-Hamiltonian in form of Eq.(18) with Hamiltonian functions  $H_1$  and  $H_2$  in Eqs.(36) and (38), and Poisson vectors  $\mathbf{J}_1$  and  $\mathbf{J}_2$  in Eqs.(37) and (39), respectively.

### 3.3 The $T$ -system

Explicitly,  $T$ -system is given by

$$\begin{aligned} \dot{x} &= \alpha(y - x), \\ \dot{y} &= (\gamma - \alpha)x - \alpha xz, \\ \dot{z} &= xy - \beta z, \end{aligned} \quad (40)$$

with  $\alpha, \beta, \gamma$  being real parameters for  $\alpha$  being non-zero [37]. When  $\beta = 2\alpha$ , the system (40) admits a time-dependent first integral

$$H_1 = e^{2\alpha t}(x^2 - 2\alpha z). \quad (41)$$

Under the change of variables

$$u = xe^{\alpha t}, \quad v = y, \quad w = ze^{2\alpha t} \quad (42)$$

the  $T$ -system (40) is transformed to a non-autonomous system

$$\dot{u} = \alpha ve^{\alpha t}, \quad \dot{v} = (\gamma - \alpha)ue^{-\alpha t} - \alpha uwe^{-3\alpha t}, \quad \dot{w} = uve^{\alpha t} \quad (43)$$



whereas the first integral  $H_1$  in Eq.(41) turns out to be time independent function

$$H_1 = u^2 - 2\alpha w. \quad (44)$$

The Jacobi's last multiplier for the set of equations (43) is  $M = 1$ . Hence, we compute Poisson vector field

$$\mathbf{J}_1 = \nabla H_1 = (2u, 0, -2\alpha) \quad (45)$$

and Hamiltonian function

$$H_2 = \frac{1}{2\alpha} \left[ \frac{1}{8} e^{-3\alpha t} u^4 + \frac{1}{2} (\gamma - \alpha) e^{-\alpha t} u^2 - \frac{\alpha}{2} u^2 w e^{-3\alpha t} \right] - \frac{1}{4} v^2 e^{\alpha t}. \quad (46)$$

Note that, the Hamiltonian function  $H_2$  in Eq.(46) is non-autonomous hence not an integral invariant of the motion. Using  $H_2$ , we construct a second Poisson vector

$$\mathbf{J}_2 = -\nabla H_2 = \left( -\frac{1}{4\alpha} u^3 e^{-3\alpha t} + \frac{1}{2\alpha} (\gamma - \alpha) u e^{-\alpha t} + \frac{1}{2} u w e^{-3\alpha t}, \frac{v}{2} e^{\alpha t}, \frac{1}{4} u^2 e^{-3\alpha t} \right). \quad (47)$$

The  $T$ -system given in Eq.(43) can be recasted as a bi-Hamiltonian form as in Eq.(18) where the Hamiltonian functions  $H_1$  and  $H_2$  are in Eqs.(44) and (46), and Poisson vectors  $\mathbf{J}_1$  and  $\mathbf{J}_2$  are in Eqs.(45) and (47), respectively.

### 3.4 The Chen system

Chen system

$$\begin{aligned} \dot{x} &= \alpha(y - x), \\ \dot{y} &= (\gamma - \alpha)x + \gamma y - xz, \\ \dot{z} &= xy - \beta z, \end{aligned} \quad (48)$$

is a chaotic system obtained by adding a state feedback to the second equation of the Lorenz system [4, 38]. The Chen system (48) is topologically inequivalent to the Lorenz system and may be considered as the dual of the Lorenz system in the sense defined by Vaněček and Čelíkovský [5]. When  $\beta = 2\alpha$  it admits a first integral

$$H_1 = e^{2\alpha t} (x^2 - 2\alpha z). \quad (49)$$

Employing the change of variables

$$u = x e^{\alpha t}, \quad v = y e^{-\gamma t}, \quad w = z e^{2\alpha t}, \quad (50)$$

the system (48) may be expressed as

$$\dot{u} = \alpha v e^{(\gamma + \alpha)t}, \quad \dot{v} = (\gamma - \alpha) u e^{-(\gamma + \alpha)t} - \alpha u w e^{-(3\alpha + \gamma)t}, \quad \dot{w} = u v e^{(\gamma + \alpha)t}. \quad (51)$$

Under the change of coordinates presented in Eq.(50), the time dependent first integral  $H_1$  in Eq.(49) becomes autonomous

$$H_1 = u^2 - 2\alpha w. \quad (52)$$

For the system (51), the Jacobi's last multiplier can be taken as the unity, hence the corresponding Poisson vector  $\mathbf{J}_1$  is

$$\mathbf{J}_1 = \nabla H_1 = (2u, 0, -2\alpha) \quad (53)$$

while the Hamiltonian is

$$H_2 = \frac{1}{2\alpha} \left[ \frac{1}{8\alpha} e^{-(3\alpha+\gamma)t} u^4 + \frac{1}{2} (\gamma - \alpha) e^{-(\gamma+\alpha)t} u^2 - \frac{1}{2} u^2 w e^{-(3\alpha+\gamma)t} \right] - \frac{1}{4} v^2 e^{(\gamma+\alpha)t}. \quad (54)$$

Using  $H_2$ , we compute the second Poisson matrix

$$\mathbf{J}_2 = -\nabla H_2 = \left( \frac{e^{-(\gamma+\alpha)t}}{2\alpha} \left( e^{-2\alpha t} u \left( w - \frac{u^2}{2\alpha} \right) - (\gamma - \alpha) u \right), \frac{e^{(\gamma+\alpha)t}}{2} v, \frac{e^{-(3\alpha+\gamma)t}}{4\alpha} u^2 \right). \quad (55)$$

Note that, this Poisson vector is similar to the one in Eq.(47) but it has a finite value for large  $t$  and  $\gamma = -\alpha$ . By substituting the Poisson vectors  $\mathbf{J}_1$  and  $\mathbf{J}_2$  in Eqs.(53) and (55), and Hamiltonian functions  $H_1$  and  $H_2$  in Eqs.(52) and (54), into Eq.(18), we arrive bi-Hamiltonian structure of Chen system in Eq.(51).

In [19], a system

$$\dot{x} = \alpha(y - x), \quad \dot{y} = (\alpha - \gamma)x + \gamma y + \lambda xz, \quad \dot{z} = xy - \beta z \quad (56)$$

has been introduced by rescaling the cross term  $xz$  in the second equation of the Chen system (48) by a real coefficient  $\lambda$ . We shall show that the chaotic system (56) is also bi-Hamiltonian. For  $\alpha = \beta = -\gamma$ , it admits a time dependent integral

$$F_1 = e^{2\alpha t} (x^2 - y^2/2 + \lambda z^2/2). \quad (57)$$

Upon making use of the transformation  $(x, y, z) \rightarrow (u, v, w)$  given by

$$u = x e^{\alpha t}, \quad v = y e^{\alpha t}, \quad zw = z e^{\alpha t}, \quad (58)$$

the system (48) turns out to be

$$\dot{u} = \alpha v, \quad \dot{v} = 2\alpha u + \lambda u w e^{-\alpha t}, \quad \dot{w} = u v e^{-\alpha t}. \quad (59)$$

The system (59) is bi-Hamiltonian with Hamiltonian functions

$$F_1 = u^2 - v^2/2 + \lambda w^2/2, \quad F_2 = \alpha w - u^2 e^{-\alpha t}/2 \quad (60)$$

and the corresponding Poisson vectors

$$\mathbf{J}_1 = \nabla F_1 = (2u, -v, \lambda w), \quad \mathbf{J}_2 = -\nabla F_2 = (u e^{-\alpha t}, 0, -\alpha), \quad (61)$$

respectively.

### 3.5 The Qi system

Chaotic Qi system

$$\begin{aligned} \dot{x} &= \alpha(y - x) + yz, \\ \dot{y} &= \gamma x - xz - y, \\ \dot{z} &= xy - \beta z \end{aligned} \quad (62)$$

involves the addition of a cross product nonlinear term to the first equation of the Lorenz system while retaining the linear feedback term in the second equation of the Chen system

(48), see [32]. When the parameters of the Qi system (62) satisfies  $\alpha = \beta = 1$ , it admits a time-dependent first integral

$$H_1 = e^{2t} (\gamma x^2 - y^2 - (\gamma + 1)z^2). \quad (63)$$

After the change of variables  $(x, y, z)$  to  $(u, v, w)$  given by

$$u = xe^t, \quad v = ye^t, \quad w = ze^t, \quad (64)$$

Qi system (62) turns out to be non-autonomous

$$\dot{u} = v + vwe^{-t}, \quad \dot{v} = \gamma u - uwe^{-t}, \quad \dot{w} = uve^{-t} \quad (65)$$

whereas the integral in Eq.(63) becomes time independent

$$H_1(u, v, w) = \gamma u^2 - v^2 - (\gamma + 1)w^2. \quad (66)$$

The Jacobi last multiplier  $M = 1$  and the Poisson vector

$$\mathbf{J}_1 = \nabla H_1 = (2\gamma u, -2v, -2(\gamma + 1)w) \quad (67)$$

with the Hamiltonian function

$$H_2 = \frac{1}{2}w - \frac{1}{4(\gamma + 1)}(u^2 + v^2)e^{-t}. \quad (68)$$

The second Poisson vector is

$$\mathbf{J}_2 = -\nabla H_2 = \left( \frac{u}{2(\gamma + 1)}e^{-t}, \frac{v}{2(\gamma + 1)}e^{-t}, -\frac{1}{2} \right). \quad (69)$$

By this we have achieved to write the bi-Hamiltonian structure of the system (65) where the Hamiltonian functions  $H_1$  and  $H_2$  are in Eqs.(63) and (68), and Poisson vectors in  $\mathbf{J}_1$  and  $\mathbf{J}_2$  are in Eqs.(67) and (69), respectively.

## 4 Discussions and outlook

In this study, we have presented bi-Hamiltonian structures of three dimensional chaotic autonomous Lü, modified Lü,  $T$ , Chen and Qi systems for some certain parameters that they involve. While achieving this, two characteristics of three dimensional Hamiltonian systems were essential. The first one is that, in three dimensions, a Hamiltonian system is mutually bi-Hamiltonian [13], and the second one is that, in three dimensions, a dynamical system is bi-Hamiltonian if and only if PDE in Eq.(24) has a non-trivial solution [8].

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